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Quantum Group Symmetry of the Hubbard Model

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Quantum group symmetry is shown to exist for the Hubbard model. It is extended to include infinitesimally deformed phonons. A simplified version of Alam's model is generalized to include phonons and is shown to have quantum group symmetry.

The Hubbard model [5] is much studied since it models strongly interacting electron systems. Sometimes it is also used to study high-temperature superconductivity. Therefore studying its symmetries and extending them (if possible) is an important task. Several authors [4, 8] have shown that it has a SO(4) $= SU(2) \times SU(2)/Z_2$ symmetry at half-filling. It it was also shown that it has $SU_q(2) \times SU_q(2)/Z_2$ quantum-group symmetry [6] and it is argued that this symmetry can be obtained from an ordinary symmetry via a "twist," which is not an equivalence transformation. We prefer to work directly with the quantum–group symmetry for reasons that will be mentioned later. Recently it has been argued that quantuam-group symmetry is related to high-temperature superconductivity [2]. Here the model of Montrosi and Rasetti [6] is extended to include infinitesimally deformed [1] phonons so both fermionic and bosonic sectors of the model are deformed. In addition, a simplified version of the stripes model of Alam and Rahman [2] is extended to include phonons so that its quantum-group symmetry becomes explicit.

We begin with the standard Hubbard Hamiltonian H_1,

$$H_1 = u \sum_i n_{i\uparrow} n_{i\downarrow} - \mu \sum_{i,\sigma} n_{i\sigma} + l \sum_{\langle i,j\rangle,\sigma} a^\dagger_{j\sigma} a_{i\sigma}, \qquad n_{i\sigma} = a^\dagger_{i\sigma} a_{i\sigma} \tag{1}$$

where a_w, $a^\dagger_{i\sigma}$ are the fermionic operators with spin $\sigma \in \{\uparrow, \downarrow\}$ at site i and

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$\langle i, j\rangle$ are nearest neighbor distinct sites. The generators of $SU(2) \times SU(2)/Z_2$ symmetry are

$$J_m^+ = a_\uparrow^\dagger a_\downarrow, \qquad J_m^- = (J_m^+)^\dagger, \qquad J_m^3 = n_\uparrow - n_\downarrow \tag{2}$$

$$J_s^+ = a_\uparrow^\ddagger a_\downarrow^\dagger, \qquad J_s^- = (J_s^-)^\dagger, \qquad J_s^3 = n_\uparrow + n_\downarrow - 1 \tag{3}$$

The Z_2 symmetry comes from the conjugation $a_\downarrow \leftrightarrow a_\uparrow^\dagger$. The invariance of the first two terms in (1) requires half-filling, i.e., $\mu = u/2$. To include the nonlocal (third) term, the global operators, e.g., $J_m^+ = \Sigma_i\, a_{i\uparrow}^\dagger - a_{i\downarrow}$ and similarly for the other operators in (2) and (3), are used.

The generators in (2) and (3) satisfy the $SU_q(2)$ relations

$$[J_s^+, J_s^-] = (q_n^{J_s^3} - q_n^{-J_s^3})/(q - q^{-1}), \qquad [J_s^3, J_s^\pm] = \pm 2J_s^\pm \tag{4}$$

and similarly for the magnetic operators. The proof uses $(J^3)^3 = J^3$. The coproduct of the quantum group is defined by

$$\Delta_q(J^\pm) = J^\pm \otimes q^{-J^3/2} + q^{J^3/2} \otimes J^\pm, \qquad \Delta_q(J_3) = 1 \otimes J^3 + J^3 \otimes 1$$

To include the nonlocal term in the quantum symmetry, the Hamiltonian H_1 in (1) can be modified to include phonons [6]:

$$\begin{aligned} H_2 = {} & u \sum_i n_{i\uparrow} n_{i\downarrow} - \mu \sum_{i,\sigma} n_{i\sigma} - \boldsymbol{\lambda} \cdot \sum_{i,\sigma} n_{i,\sigma} \mathbf{x}_i \\ & + \sum_i [(\mathbf{p}_i^2)/2m + 1/2 m\omega^2 \mathbf{x}_i^2] + \sum_{(i,j),\sigma} T_{i,j} a_{ij,\sigma}^\dagger a_{i\sigma} + \text{h.c.} \end{aligned} \tag{5}$$

where $\mathbf{x}_i$ is the displacement of an ion, $\mathbf{p}_i$ is the corresponding momentum, and

$$\begin{aligned} T_{i,j} &= t \exp[i\boldsymbol{\kappa} \cdot (\mathbf{p}_i - \mathbf{p}_j)] \\ \tilde{J}^\pm &= J^\pm \exp[\mp 2i\boldsymbol{\lambda} \cdot \mathbf{p}/hm\omega^2] \end{aligned} \tag{6}$$

The Hamiltonian H_2 is invariant under $SU_q(2) \times SU_q(2)/Z_2$ provided that

$$\mu = u/2 - (\boldsymbol{\lambda}^2)/m\omega^2, \qquad \boldsymbol{\lambda} = hm\omega^2 \boldsymbol{\kappa} \tag{7}$$

So far the quantum symmetry is for the fermionic sector, and therefore it is proposed here to use infinitesimally deformed [1] phonons instead of ordinary ones. They are defined by the relations

$$\begin{aligned} [x, p] &= ih + i\varepsilon H_0 \\ [H, p] &= 1/2 hm\omega^2 x \\ [H, x] &= -ph/2m \\ H &= H_0 - \varepsilon H_0^2 \\ H_0 &= \mathbf{p}^2/2m + m\omega^2 \mathbf{x}^2/2 \end{aligned} \tag{8}$$

The bosonic deformation parameter is $q_b \simeq 1 + \varepsilon$ and one linearizes in ε. The new Hamiltonian is

$$H_3 = H_2 - \varepsilon \sum_i (\mathbf{p}_i^2/2m + 1/2m\omega^2\mathbf{x}_i^2)^2$$

The symmetry requires

$$\boldsymbol{\lambda} = hm\omega^2\boldsymbol{\kappa}$$
$$\mu = u/2 - \boldsymbol{\lambda}^2/m\omega^2[1 + \varepsilon \sum_i (1/2m\mathbf{p}_i^2 + 1/2m\omega^2\mathbf{x}_i^2)] \tag{9}$$

Alam and Rahman [2] proposed a model for stripes in the form of two interacting fermionic chains. Here a simplified version of their model is studied and generalized to include phonons. The Hamiltonian is

$$H_3 = H_2(a_{i\sigma}, a_{i\sigma}^\dagger, u, \mu, m, \omega, \boldsymbol{\lambda}, \mathbf{p}_i, \mathbf{x}_i, \boldsymbol{\kappa}) + H_2(b_{i\sigma}, b_{i\sigma}^\dagger, u', \mu', m, \omega, \boldsymbol{\lambda}, \mathbf{p}_i, \mathbf{x}_i, \boldsymbol{\kappa}) + \zeta \sum_i n_{i\sigma}n'_{i\sigma} \tag{10}$$

where $n'_{i\sigma} = b_{i\sigma}^\dagger$.

The quantum symmetry requires

$$\mu = u/2 - \boldsymbol{\lambda}^2/m\omega^2, \qquad \mu' = u'/2 - \boldsymbol{\lambda}^2/m\omega^2, \qquad \boldsymbol{\lambda} = hm\omega^2\boldsymbol{\kappa} \tag{11}$$

It is interesting to study the infinitesimally deformed fermionic oscillators and their possible relevance to high-temperature superconductivity. The deformed fermionic algebra is defined by

$$\{b_{i\sigma}^\dagger, b_{j\tau}\} = q^N\delta_{ij}\delta_{\sigma\tau}, \qquad [N, b_{i\sigma}^\dagger] = b_{i\sigma}^\dagger, \qquad [N, b_{j\tau}] = -b_{j\tau} \tag{12}$$

where $\{A, B\} = AB + BA$ and $[A, B] = AB - BA$. Let $q = 1 + \epsilon$ and linearize in ϵ. One gets

$$N \simeq \sum_\sigma (b_\sigma^\dagger b_\sigma) + \epsilon b_\uparrow^\dagger b_\downarrow^\dagger b_\downarrow b_\uparrow \tag{13}$$

Since the Hamiltonian is typically proportional to the number operator N, it should contain a quartic term of the form (13). This term is quite similar to the one used by Anderson's group [3] to explain high-temperature superconductivity. Their Hamiltonian is

$$H = -\sum_k T(k)(b_{k\uparrow}^{(1)\dagger}b_{-k\downarrow}^{(1)\dagger}b_{-k\downarrow}^{(2)}b_{k\uparrow}^{(2)} + \text{h.c.})$$

where (1) superscripts and (2) represent different CuO layers or O chains.

There are two interesting features of the quantum symmetry of the Hubbard model. The first is that phonons can be included naturally without breaking the symmetry. This is expected to be relevant since some high-temperature superconductivity materials show a nonzero isotope effect, which indicates that phonons should not be negligible. The second feature is that both magnetic and superconducting symmetries are included. Again this is expected to be relevant to high-temperature superconductivity. These are strong motivations to consider quantum groups as relevant to high-temperature superconductivity, at least as a starting point. It is hoped that this work, together with others, is a step in the right direction.

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